Morning Madness - Solution

The answer is $\det(A!) = \frac{\mu!\lambda!}{|\mu-\lambda|!}$
If $\lambda = \mu$, then $A = \lambda I$ and by definition of the factorial we find $\det(A!) = \lambda!^2$ as wanted. Now assume without loss of generality that $\lambda > \mu$ (of course, keep this in mind while programming!). Let $B_n = A - nI$ and take n such that $B = B_n$. Then we have

$$0 = \det(B) = \det(A - nI),$$

so n is an eigenvalue of A. As the eigenvalues of A are λ, μ, we have that $n = \lambda$ or $n = \mu$. Clearly, these values of n satisfy $\det(B) = 0$. Now as n is the smallest integer such that $\det(B) = 0$ and $\mu < \lambda$ we have that $n = \mu$.
Finally, we calculate $\det(A!)$. By the multiplicity property of the determinant, this equals:

$$\begin{aligned}
\det(A!) &= \det(A \cdot (A - I) \cdots (B + I)) \\
&= \det(A) \cdot \det(A - I) \cdots \det(B + I) \\
&= (\lambda \cdot \mu) \cdot ((\lambda - 1) \cdot (\mu - 1)) \cdots ((\lambda - \mu + 1) \cdot 1) \\
&= \frac{\lambda!\mu!}{(\lambda - \mu)!}.
\end{aligned}$$

This is exactly what we wanted to find. $\square$